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A Remark on a Theorem of W. E. H. Berwick

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Abstract. We indicate and fill a gap in a theorem of W. E. H. Berwick concerning the computation of the fundamental units in a semireal biquadratic field.

One of the standard methods for finding a pair of fundamental units in algebraic number fields that have two fundamental units is that of Berwick [1] (cf. the Introduction of [4]). Berwick's method provides a simple and rather easily applicable algorithm for the search of such units, especially for cubic and quartic fields. For example, in many cases the necessary computations can be made by hand, as it is seen, e.g., in the numerical applications given by Berwick himself. In other cases, a pocket calculator is sufficient (see, e.g., [2], [5], [6]). Of course, the use of a computer in some cases (as in [3] for example) is indispensable, but a personal computer may prove to be quite satisfactory (see, e.g., [7], [8]).

The purpose of the present note is to indicate and fill a gap in [1] that occurs in the case of a semireal biquadratic field K, i.e., in the case of a field K that is generated over $\mathbf{Q}$ by an element

$$(1) \qquad \begin{aligned} &\sqrt{a+b\sqrt{m}}\,, \qquad a,b,m \in \mathbf{Z},\ m>0,\ \sqrt{m}\notin \mathbf{Q},\\ &a+b\sqrt{m}>0, \qquad a-b\sqrt{m}<0. \end{aligned}$$

For any $\alpha \in K$ we denote by α', α'', $\bar{\alpha}''$ its algebraic conjugates (α' is real and α'', $\bar{\alpha}''$ are complex-conjugates).

The result of Berwick [1, Section 7] is conveniently formulated here as follows:

Let R be an order of K containing the ring of integers of $\mathbf{Q}(\sqrt{m})$. Then the set

$$E = \{\varepsilon\colon \varepsilon \text{ unit of } R,\ \varepsilon>1,\ |\varepsilon'|<1,\ |\varepsilon''|\leqslant 1\}$$

is a discrete nonempty set. Let ε_1 be the minimum element of E and $\iota>1$ the fundamental unit of $\mathbf{Q}(\sqrt{m})$. Then, $\varepsilon_1\varepsilon_1' = \pm\iota$ or ± 1. (A) *If $\varepsilon_1\varepsilon_1'=\pm\iota$, then ε_1, ι is a pair of fundamental units in R.* (B) *If $\varepsilon_1\varepsilon_1'=\pm 1$, then ε_1, ι or ε_1, $\sqrt{\iota}$ is a pair of fundamental units in R, according as $\sqrt{\iota}\notin R$ or $\sqrt{\iota}\in R$.*

Remark. In [1] only maximal orders (i.e., the rings of integers) of the various fields are considered. However, it is straightforward to see that the arguments in [1] are also valid if the maximal orders are replaced by orders containing the integers of $\mathbf{Q}(\sqrt{m})$ (a simple but important fact is that for such orders R of K we have $R'=R$, where $R'=\{\alpha'\colon \alpha\in R\}$).

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We observe that in case (B) a distinction should be made between the subcases $\iota < \varepsilon_1$ and $\varepsilon_1 < \iota$. Moreover, in the second subcase one should examine whether or not $\sqrt{\iota\varepsilon_1}$ belongs to R. However, it is not difficult to show by a slight modification of Berwick's arguments the following

THEOREM. *Let $K, R, \varepsilon_1, \iota$ be as before. Then* (A) *is valid, but* (B) *should be replaced by the following*:

(B_1) *If $\varepsilon_1\varepsilon_1' = \pm 1$ and either* (i) *$\iota < \varepsilon_1$ or* (ii) *($\varepsilon_1 < \iota$ and $\sqrt{\iota\varepsilon_1} \notin R$), then ε_1, ι or $\varepsilon_1, \sqrt{\iota}$ is a pair of fundamental units in R, according as $\sqrt{\iota} \notin R$ or $\sqrt{\iota} \in R$.*

(B_2) *If $\varepsilon_1\varepsilon_1' = \pm 1$ and $\varepsilon_1 < \iota$ and $\sqrt{\iota\varepsilon_1} \in R$, then $\iota, \sqrt{\iota\varepsilon_1}$ is a pair of fundamental units in R.*

The fact that in case (B) of Berwick no distinction is made between (B_1) and (B_2) may lead to wrong conclusions, as the following counterexample shows:

Consider the field $K = \mathbf{Q}(\sqrt{47 + 8\sqrt{35}})$. In this field, the unit

$$\xi = \tfrac{1}{2}\left(4 + \sqrt{35} + \sqrt{47 + 8\sqrt{35}}\right)$$

satisfies the equation

$$\xi^4 - 8\xi^3 - 17\xi^2 - 8\xi + 1 = 0.$$

The remaining conjugates of ξ are

$$\xi' = \tfrac{1}{2}\left(4 + \sqrt{35} - \sqrt{47 + 8\sqrt{35}}\right) = \xi^{-1},$$
$$\xi'' = \tfrac{1}{2}\left(4 - \sqrt{35} + \sqrt{47 - 8\sqrt{35}}\right),\ \bar{\xi}''.$$

It is not difficult to see that the order $\mathbf{Z}[1, \sqrt{35}, \xi, \xi\sqrt{35}] = R$ is the ring of integers of K (although this is not necessary). The typical element ε of R has the form

$$\varepsilon = x + y\sqrt{35} + z\xi + w\xi\sqrt{35}, \qquad x, y, z, w \in \mathbf{Z},$$

so that the relations

$$1 < \varepsilon < 10, \quad |\varepsilon'| < 1, \quad |\varepsilon''| \leqslant 1$$

imply a system of linear inequalities in x, y, z, w:

(2) $\quad 1 < \varepsilon < 10, \quad -1 < \varepsilon' < 1, \quad -1 \leqslant \mathrm{Re}(\varepsilon'') \leqslant 1, \quad -1 \leqslant \mathrm{Im}(\varepsilon'') \leqslant 1.$

Some easy calculations show that the only ε satisfying (2), and being also a unit, is obtained for $x = 0$, $y = 0$, $z = 1$, $w = 0$. Thus, in Berwick's notation,

$$\varepsilon_1 = \xi \quad \text{and} \quad \varepsilon_1\varepsilon_1' = 1.$$

On the other hand,

$$\iota = 6 + \sqrt{35},$$

and it is an easy exercise to show that $\sqrt{\iota} \notin K$, so that by Berwick's result (B), ι, ε_1 should be a pair of fundamental units for R. Note, however, that

$$\iota\varepsilon_1 = \tfrac{1}{4}\left(6 + \sqrt{35} + \sqrt{47 + 8\sqrt{35}}\right)^2 = (1 + \xi)^2,$$

which means that the unit $\sqrt{\iota\varepsilon_1} = 1 + \xi \in R$ does not belong to the unit group generated by ι and ε_1. Thus, ι, ε_1 are not a pair of fundamental units in R, contrary to Berwick's result (B). In fact, we are in case (B_2) (note that $\varepsilon_1 < \iota$) and a pair of fundamental units is $\iota, 1 + \xi(= \sqrt{\iota\varepsilon_1})$.

Remark. Ray Steiner and the referee have pointed out to me that in the case of the field $\mathbf{Q}(\sqrt[4]{6})$, the value of ε_1 given in page 372 of Berwick's paper, namely

$$\varepsilon_1 = 11425 + 7300\vartheta + 4664\vartheta^2 + 2980\vartheta^3,$$

is not correct. In fact, if one looks at Ljunggren's paper on biquadratic fields (1936), one finds that

$$\varepsilon_1 = (53 + 34\vartheta + 22\vartheta^2 + 14\vartheta^3)^2.$$

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